Math 3235 Probability Theory

3/30/23

X $\quad \mu = 2 \quad \mathrm{var}(X) = \sigma^2 = 4$

$$Z = \frac{X-2}{2} \approx N(0,1)$$

$$\mathbb{P}(X < 0) = \mathbb{P}\left(\frac{X-2}{2} < -1\right) =$$

$$= \Phi(-1)$$

Find δ such that

$$\mathbb{P}(2-\delta \leq X \leq 2+\delta) = 0.95$$

$$\parallel$$

$$\mathbb{P}\left(-\frac{\delta}{2} \leq \frac{X-2}{2} \leq \frac{\delta}{2}\right) = 0.95$$

$$P\left(Z < -\frac{\delta}{2}\right) = \mathbb{P}\left(Z > \frac{\delta}{2}\right)$$

$$\bar{\Phi}\left(-\frac{\delta}{2}\right) = 0.025$$

$$\bar{\Phi}\left(-z_\alpha\right) = \alpha$$

z_α critical value

$$z_{0.025} = 1.96$$

$$\delta = 2\, z_{0.025} = 3.92$$

$$f_X(x) = \begin{cases} 4x e^{-2x} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$f_{Y|X}(y|x) = \begin{cases} \frac{1}{x} & 0 \le y \le x \\ 0 & \text{otherwise} \end{cases}$$

j. p. d. f. of X, Y

$$f(x,y) = \begin{cases} 4e^{-2x} & x > y > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$f_Y(y) = \int_y^{\infty} 4e^{-2x}\,dx = 2e^{-2y}$$

$y > 0$

$$f_{X|Y}(x|y) = \frac{4e^{-2x}}{2e^{-2y}} = 2e^{-2(x-y)}$$

$x > y > 0$

$$\mathbb{P}\left(Y > X/2\right)$$

$$f_{Y|X}(y|x) = \frac{1}{x} \qquad 0 \leq y \leq x$$

$$\mathbb{P}\left(Y > \frac{X}{2} \,\middle|\, X = x\right) = \frac{1}{2}$$

$$\mathbb{P}\left(Y > \frac{X}{2}\right) = \int_{-\infty}^{\infty} \mathbb{P}\left(Y > \frac{X}{2} \,\middle|\, X = x\right) f_X(x)\,dx$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} f_X(x)\,dx = \frac{1}{2}$$

$$\mathbb{P}\left(Y > X/_2\right) = \iint_{0<y<\frac{x}{2}} 4e^{-2x}\,dx\,dy =$$

$$\int_0^{\infty}\int_0^{x/_2} 4e^{-2x}\,dy\,dx =$$

$$\int_0^{\infty} 2xe^{-2x}\,dx = \frac{1}{2}$$

$$U = Y \qquad V = X - Y$$

$$f_{U,V}(u,v) = 4e^{-2(u+v)} \qquad u,v > 0$$

U V are exp par 2

independent.

$X \quad Y$ Normal and indep

$N(0, 1)$

$U = \frac{1}{\sqrt{2}}(X + Y) \qquad \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$V = \frac{1}{\sqrt{2}}(X - Y) \qquad \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$

joint p.d.f. of X and Y.

$$\begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}$$

$$A A^T = Id$$

$$X = \frac{1}{\sqrt{2}}(U + V)$$

$$Y = \frac{1}{\sqrt{2}}(U - V)$$

$$\left| \det\left(\frac{\partial(x,y)}{\partial(u,v)} \right) \right| = |\det A| = 1$$

$$\frac{1}{2\pi} e^{-\frac{x^2+y^2}{2}} = \frac{1}{2\pi} e^{-\frac{u^2+v^2}{2}}$$

$$f_{U,V}(u,v) = \frac{1}{2\pi} e^{-\frac{u^2+v^2}{2}}$$

X exponential of par λ

N of atoms each decays with exp dist.

half life The Time needed To have $N/2$ atoms still undecayed : $t_{1/2}$

$$\mathbb{P}(X \leq t_{1/2}) = \frac{1}{2}$$

$$1 - e^{-\lambda t_{1/2}} = \frac{1}{2}$$

$$\lambda t_{1/2} = \ln 2$$

$$t_{1/2} = \frac{\ln 2}{\lambda}$$

$t_{1/4}$ Time needed To have only $\frac{N}{4}$ atoms left : $t_{1/4}$

$$\mathbb{P}\left(X \leq t_{1/4} \right) = \frac{3}{4}$$

$$e^{-\lambda t_{1/4}} = \frac{1}{4}$$

$$t_{1/4} = 2 t_{1/2}$$

N D_1 ... D_N

D_i Bernoulli and indep.

$D_i = 1$ prob $\frac{1}{4}$

$$T_0 = \sum_{i=1}^{N} D_i$$

Cauchy - ~~Black~~ Schwarz Inequality

$$\mathbb{E}(UV)^2 \leq \mathbb{E}(U^2)\,\mathbb{E}(V^2)$$

$$W = U + sV$$

$$\mathbb{E}(W^2) \geq 0$$

$$\mathbb{E}(U^2) + 2s\,\mathbb{E}(UV) + s^2\,\mathbb{E}(V^2) \geq 0$$

$$f(s) = c + 2s\,b + s^2\,a$$

$$f'(s) = 2b + 2as$$

$$f'(s) = 0 \implies s = -\frac{b}{a}$$

$$c - 2\frac{b^2}{a} + \frac{b^2}{a} = c - \frac{b^2}{a} \geq 0$$

$$b^2 \leq ca$$

$$\mathbb{E}(UV)^2 \leq \mathbb{E}(U^2)\,\mathbb{E}(V^2)$$

X Y r.v.

$$\operatorname{cov}(X,Y)^2 = \mathbb{E}\big((X-\mathbb{E}(X))(Y-\mathbb{E}(Y))\big)^2$$

$$\leq \mathbb{E}\big((X-\mathbb{E}(X))^2\big)\,\mathbb{E}\big((Y-\mathbb{E}(Y))^2\big) =$$

$$\operatorname{var}(X)\operatorname{var}(Y)$$

$$\rho_{X,Y} = \frac{\operatorname{Cov}(X\,Y)}{\sqrt{\operatorname{var}(X)\operatorname{var}(Y)}}$$

$$\rho_{XY}^2 \leq 1$$

There exists s such that

$$\mathbb{E}(W^2) = 0$$

$$E((U+sV)^2) = 0$$

$$U + sV \equiv 0$$

$$U = sV.$$

$$\mathbb{E}(UV)^2 = \mathbb{E}(U^2)\,\mathbb{E}(V^2)$$

$$\Downarrow$$

$$U = sV$$

U V are r.v.

$$\mathbb{E}(UV)^2 \leq \mathbb{E}(U^2)\,\mathbb{E}(V^2)$$

Equality holds iff $U = sV$.